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ON BANACH SPACES WHOSE DUAL BALLS ARE NOT WEAK* SEQUENTIALLY COMPACT

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ABSTRACT

THEOREM 1. Let X be a Banach space. (a) If X^* has a closed subspace in which no normalized sequence converges weak* to zero, then l_1 is isomorphic to a subspace of X. (b) If X^* contains a bounded sequence which has no weak* convergent subsequence, then X contains a separable subspace whose dual is not separable.

The common feature in the proofs of (a) and (b) of Theorem 1 is a diagonalization argument similar to that of Nissenzweig [5]. Nissenzweig's main result, that every conjugate Banach space contains a normalized sequence which weak* converges to zero, is a consequence of Corollary 1. (This result was also proved by Josefson [3a].)

COROLLARY 1. *If l_1 is isomorphic to a subspace of X^*, or if the unit ball of X^* is not weak* sequentially compact, then either c_0 is isomorphic to a quotient of X or l_1 is isomorphic to a subspace of X.*

One derives Nissenzweig's result from Corollary 1 as follows: If X^* contains no normalized weak* null sequence, then the unit ball of X^* is not weak* sequentially compact. Thus either c_0 is a quotient of X, or, if l_1 embeds into X, l_2 is a quotient of X by a result of Pełczyński's [6]. At any rate, X has a separable quotient, which implies that X^* contains a normalized weak* null sequence.

In view of a theorem of Stegall [8], we can restate Theorem 1 (b) as

COROLLARY 2. *If X^* has the Radon–Nikodym property, then the unit ball of X^* is weak* sequentially compact.*

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Haydon [2] has recently given an example of a compact Hausdorff space K which is not sequentially compact such that $l_1(\Gamma)$ does not embed in $C(K)$ for any uncountable set Γ. It is not known, however, if the hypothesis of Theorem 1 (b) implies that l_1 embeds in X.

NOTATION. Unexplained notation is standard and can be found in [1] or [4].

All Banach spaces are real. For a Banach space X, B_X denotes the closed unit ball of X. If D is a subset of X, then $[D]$ is the closed linear span of the set D in X. If X and Y are Banach spaces, then $(X \oplus Y)_1$ is the Banach space $(X \times Y, \| \|)$, where $\|(x, y)\| = \|x\| + \|y\|$.

Next, let T denote the set of finite sequences of 0's and 1's. For $\varphi \in T$, $|\varphi|$ denotes the length of the sequence φ. If $\varphi, \psi \in T$, then $\varphi \geqq \psi$ if $|\varphi| \geqq |\psi|$ and the first $|\psi|$ terms of φ form the sequence ψ. If $|\psi| = n$ and $i = 0$ or 1, then ψ, i is the unique sequence of length $n + 1$ whose first n terms form the sequence ψ. Also, let Δ denote the set of infinite sequences of 0's and 1's. Then to each $\xi \in \Delta$ we can associate a unique sequence (φ_n) in T, where for each $n \geqq 0$, the first n terms of ξ form the sequence φ_n.

PROOF OF (a) OF THEOREM 1. Let (f_n) be a normalized basic sequence in X^* such that no normalized sequence in $[f_n]$ converges weak* to zero. Then by Rosenthal's characterization of l_1-sequences [7], we can assume that (f_n) is equivalent to the usual basis of l_1.

First, we may assume that (f_n) is isometrically equivalent to the usual basis of l_1. To see this, observe that l_1 imbeds in X if and only if l_1 imbeds in $(X \oplus c_0)_1$. If (h_n) denotes the usual basis for $l_1 \cong c_0^*$, then the sequence (f_n, h_n) in $(X \oplus c_0)_1^*$ is isometrically equivalent to the usual basis of l_1 and there is no normalized sequence in $[(f_n, h_n)]$ which tends weak* to zero.

Now, let (g_n) be any sequence in a Banach space isometrically equivalent to the usual basis of l_1. Then a sequence (h_n) is called an *l_1-normalized block* of (g_n) (in short, a *block*) if $h_n = \Sigma_{i \in A_n} \alpha_i g_i$ where the A_n are pairwise disjoint finite subsets of the integers and $\Sigma_{i \in A_n} |\alpha_i| = 1$. It is clear that every block (g_n) of (h_n) is isometrically equivalent to the usual basis of l_1.

For a block (g_n) of our original sequence (f_n) in B_{X^*}, define $\delta(g_n) = \sup_x \limsup_n g_n(x)$ where the sup is taken over all $x \in B_X$. Also define

$$\varepsilon(g_n) = \inf\{\delta(h_n) : (h_n) \quad \text{is a block of} \quad (g_n)\}.$$

It is clear that $\delta(g_n) > 0$ and if (h_n) is a block of (g_n), then $\delta(h_n) \leqq \delta(g_n)$ and $\varepsilon(h_n) \geqq \varepsilon(g_n)$.

We claim that there exists a block (g_n) of (f_n) such that $\varepsilon(g_n) = \delta(g_n)$, i.e.

$\delta(h_n) = \delta(g_n)$ for any block (h_n) of (g_n). Indeed, we can pick a block (g_n^1) of (f_n) such that $\delta(g_n^1) < \varepsilon(f_n) + 2^{-1}$, and, inductively, we can pick blocks (g_n^k) of (g_n^{k-1}) such that $\delta(g_n^k) < \varepsilon(g_n^{k-1}) + 2^{-k}$. Putting $g_n = g_n^n$ for each n, we have that (g_n) is a block of (f_n) which satisfies $\delta(g_n) \geqq \varepsilon(g_n)$.

We use this sequence (g_n) to pick a sequence (ω_n) in B_X equivalent to the usual basis of l_1. Let $\delta = \delta(g_n) = \varepsilon(g_n)$ and pick $\varepsilon < \delta/2$. Then there exists $\omega_1 \in B_X$ and an infinite subset A_ϕ of N such that $\delta - \varepsilon < g_n(\omega_1)$ for $n \in A_\phi$.

Assume now that we have picked $\omega_1, \cdots, \omega_m \in B_X$ and infinite subsets A_φ of $\mathbf{N}$ for all $\varphi \in T, |\varphi| = n$, $n = 1, \cdots, m-1$ to satisfy

(1) $A_{\varphi,0} \cup A_{\varphi,1} \subset A_\varphi$ for all φ, $0 \leqq |\varphi| \leqq m-2$;

(2) $g_k(\omega_n) > \delta - \varepsilon$ if $k \in A_{\varphi,0}$ for some $\varphi, |\varphi| = n-2$; and $g_k(\omega_n) < -\delta + \varepsilon$ if $k \in A_{\varphi,1}$ for some φ, $|\varphi| = n-2$.

We want to pick $\omega_{m+1} \in B_X$ and infinite subsets A_φ *of* $\mathbf{N}$ for all $\varphi, |\varphi| = m$ to satisfy (1) and (2).

For all $\varphi, |\varphi| = m-1$, choose infinite disjoint subsets $A'_{\varphi,0}$ and $A'_{\varphi,1}$ of A_φ, and enumerate $A'_{\varphi,i} = \{n_k^{\varphi,i}\}$ for $i = 0, 1$. Consider the block of (g_k) whose kth term is

$$\frac{1}{2^m} \sum_{|\varphi|=m-1} (g_{n_k}^{\varphi,0} - g_{n_k}^{\varphi,1}).$$

Let $0 < \varepsilon' < \varepsilon/(2^{m+1}-1)$. Then there exists $\omega_{m+1} \in B_X$ so that for all but finitely many k,

$$(\alpha) \qquad \delta - \varepsilon' < \frac{1}{2^m} \sum_{|\varphi|=m-1} (g_{n_k}^{\varphi,0} - g_{n_k}^{\varphi,1})(\omega_{m+1}).$$

Also, for all but finitely many k, all $|\varphi| = m-1$ and $i = 0$ or 1,

$$(\beta) \qquad |g_{n_k}^{\varphi,i}(\omega_{m+1})| < \delta + \varepsilon'.$$

A simple calculation shows that for all k satisfying (α) and (β),

$$g_{n_k}^{\varphi,0}(\omega_{m+1}) > \delta - (2^{m+1}-1)\varepsilon'$$

and

$$g_{n_k}^{\varphi,1}(\omega_{m+1}) < -\delta + (2^{m+1}-1)\varepsilon'.$$

We will prove this for a distinct $\psi, 0$, the proof for $\psi, 1$ being the same. If $g_{n_k}^{\psi,0}(\omega_{m+1}) \leqq \delta - (2^{m+1}-1)\varepsilon'$, then

$$\delta - \varepsilon' < \frac{1}{2^m} \sum_{|\varphi|=m-1} (g_{n_k}^{\varphi,0} - g_{n_k}^{\varphi,1})(\omega_{m+1})$$

$$= 2^{-m} \Big(g_{n_k}^{\psi,0}(\omega_{m+1}) - g_{n_k}^{\psi,1}(\omega_{m+1}) + \sum_{\substack{|\varphi|=m-1 \\ \varphi \neq \psi}} (g_{n_k}^{\varphi,0}(\omega_{m+1}) - g_{n_k}^{\varphi,1}(\omega_{m+1})\Big)$$

$$< 2^{-m}(\delta - (2^{m+1}-1)\varepsilon' + (\delta + \varepsilon') + (2^m - 2)(\delta + \varepsilon'))$$
$$= 2^{-m}(2^m\delta - (2^{m+1} - 2^m)\varepsilon')$$
$$= \delta - \varepsilon',$$

which contradicts the choice of ω_{m+1}.

Put $A_{\varphi,0} = \{n_k : g_{n_k}^{\varphi,0}(\omega_{m+1}) > \delta - \varepsilon\}$ and $A_{\varphi,1} = \{n_k : g_{n_k}^{\varphi,1}(\omega_{m+1}) < -\delta + \varepsilon\}$. Since $(2^{m+1}-1)\varepsilon' < \varepsilon$, $A_{\varphi,0}$ and $A_{\varphi,1}$ are infinite subsets of $\mathbf{N}$ for all φ, $|\varphi| = m-1$. This completes the induction.

The following easy argument shows that the sequence (ω_k) is $(\delta - \varepsilon)$ equivalent to the usual basis of l_1. Let n, and scalars $t_1, \cdots, t_n$ with $t_1 \geqq 0$ be given. Let

$$\varepsilon_i = \begin{cases} 1 & \text{if} \quad t_{i+1} \geqq 0 \\ 0 & \text{if} \quad t_{i+1} < 0 \end{cases} \quad i = 1, 2, \cdots, n-1,$$

and set $\varphi = (\varepsilon_1, \cdots, \varepsilon_{n-1})$. Then if $g = g_{n_k}^{\varphi}$ for some $n_k^{\varphi} \in A_{\varphi}$,

$$g(\omega_i) > \delta - \varepsilon \qquad \text{if} \quad t_{i+1} \geqq 0$$

and

$$g(\omega_i) < -\delta + \varepsilon \quad \text{if} \quad t_{i+1} < 0.$$

Hence, $\|\sum_{i=1}^n t_i\omega_i\| \geqq \sum_{i=1}^n t_i g(\omega_i) > (\delta - \varepsilon)\sum_{i=1}^n |t_i|$, so (ω_k) is equivalent to the usual basis of l_1. This completes the proof of Theorem 1 (a).

Proof of (b) of Theorem 1. For this part of the proof, let us adopt the following conventions: We will denote infinite sequences in X^* by the capital letters M and N. A sequence M_1 is almost contained in M_2 (denoted $M_1 \subset_a M_2$) if $M_1 \backslash M_2$ is finite. For a sequence M in B_{X^*}, M' denotes the set of weak* accumulation points of M, and hull* (M') the weak* closed convex hull of M'.

For any sequence M in B_{X^*}, define $D(M) = \sup \inf_{f_0, f_1} |f_0(x) - f_1(x)|$ where the sup is taken over all M_0, $M_1 \subset_a M$, $f_i \in \text{hull}^*(M_i')$ for $i = 0, 1$, and $x \in B_X$.

The following consequence of the separation theorem will be needed.

Lemma 5. *Let N be a sequence in B_{X^*} with no weak* converging subsequence. Then if M is any subsequence of N, $D(M) > 0$.*

Proof. Pick $g_0, g_1 \in M'$ with $g_0 \neq g_1$. Let V_0, V_1 be disjoint weak* closed convex neighborhoods of g_0, g_1 respectively. For $i = 0, 1$, put $M_i = \{m \in M : m \in V_i\}$. Then both M_i are infinite, both hull*(M_i') are weak* compact and convex, and hull*$(M_0') \cap$ hull*$(M_1') = \varnothing$. The separation theorem yields an $x \in B_X$ and a $\delta > 0$ such that $f_0(x) - \delta > f_1(x)$ for all $f_i \in \text{hull}^*(M_i')$, $i = 0, 1$, and this implies $D(M) > \delta$. Q.E.D.

Again, let N be a sequence in B_{X^*} with no weak* converging subsequence. For each $\varphi \in T$, pick a subsequence N_φ of N, and infinite $N_{\varphi,i} \subset_a N_\varphi$ $(i = 0, 1)$, and $x_\varphi \in B_X$ such that $(f_0 - f_1)(x_\varphi) \geqq \frac{1}{2} D(N_\varphi)$ for any $f_i \in \text{hull}^*(N'_{\varphi,1})$, $i = 0, 1$. (This selection can be accomplished by induction on sequences of length n.)

We claim that there exists a $\varphi_0 \in T$ and a $\delta > 0$ such that, for all $\psi \geqq \varphi_0$, $D(N_\psi) \geqq 2\delta$. If not, there exists a sequence $\varphi_1, \varphi_2, \cdots$ in T such that $\varphi_1 \leqq \varphi_2 \leqq \cdots$ and $D(N_{\varphi_k}) < 1/k$ for all k. Let $M = \{m_1, m_2, \cdots\}$ be a sequence of distinct elements in N such that $m_k \in \bigcap_{j=1}^{k} N_{\varphi_j}$ for each k. Then $M \subset_a N_{\varphi_k}$ for each k, and by Lemma 4, $D(M) > 0$. But $D(N_{\varphi_k}) \geqq D(M)$ for all k, which is a contradiction.

So, by reindexing if necessary, we can assume that $\varphi_0 = \varnothing$, i.e., that for all $\varphi \in T$, $|(f_0 - f_1)(x_\varphi)| \geqq \delta$ if $f_i \in \text{hull}^*(N'_{\varphi,i})$, $i = 0, 1$. Let $X_0 = [\{x_\varphi : \varphi \in T\}]$. Since T is countable, X_0 is separable. To show that X_0^* is nonseparable, for each $\xi \in \Delta$ pick one element $f_\xi \in \bigcap_{j=0}^{\infty} N'_{\varphi_j}$ where (φ_j) is the unique sequence in T generated by ξ. (This intersection is non-empty by compactness.)

Now, let $\hat{f}_\xi = f_{\xi \mid X_0}$. Given distinct $\xi, \eta \in \Delta$, let (φ_j) and (ψ_j) be the sequences in T corresponding to ξ, η respectively. Let $j_0 = \max\{j : \varphi_j = \psi_j\}$ and $\varphi = \varphi_{j_0}$. Then without loss of generality, we can assume that $f_\xi \in N'_{\varphi,0}$ and $f_\eta \in N'_{\varphi,1}$. But then, $\|\hat{f}_\xi - \hat{f}_\eta\| \geqq (f_\xi - f_\eta)(x_\varphi) \geqq \delta$. Since Δ is uncountable, this shows that X_0^* is nonseparable. Q.E.D.

Proof of Corollary 1. By Rosenthal's theorem [7], if B_{X^*} is not weak* sequentially compact, then l_1 embeds into X^*. Now if l_1 does not embed into X, then by Theorem 1 (a), B_{X^*} contains a weak* null sequence which is equivalent to the unit vector basis of l_1. Thus by remark III.1 of [3], c_0 is isomorphic to a quotient of X.

Added in proof. E. Odell and the first named author have constructed a Banach space not containing l_1 whose dual ball is not weak* sequentially compact.

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